Direct Proof – Rational Numbers

Lecture 14
Section 4.2

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Outline

Rational Numbers

Definition (Rational Number)

A rational number is a real number that can be represented as the quotient of two integers. That is, a real number r is rational if

$$\exists a, b \in \mathbb{Z}, r = \frac{a}{b}.$$

Properties of Rational Numbers

Theorem

The sum of two rational numbers is a rational number.

Proof.

- Let r and s be rational numbers.
- Then there exist integers a, b, c, and d such that $r = \frac{a}{b}$ and $s = \frac{c}{d}$.
- Then

$$\begin{aligned} r + s &= \frac{a}{b} + \frac{c}{d} \\ &= \frac{ad}{bd} + \frac{bc}{bd} \\ &= \frac{ad + bc}{bd}. \end{aligned}$$

□

Properties of Rational Numbers

Proof.

- $ad + bc$ and bd are integers.
- Furthermore, $bd \neq 0$ because $b \neq 0$ and $d \neq 0$.
- Therefore, $r + s$ is a rational number.

□

Outline

Definition (Harmonic Mean)

Let x and y be two nonzero real numbers with $y \neq -x$. The harmonic mean $h(x, y)$ of x and y is the reciprocal of the average of their reciprocals. That is,

$$h(x, y) = \frac{1}{\left(\frac{\frac{1}{x}+\frac{1}{y}}{2}\right)}.$$

- Note that

$$\frac{1}{h(x, y)} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)$$

and that

$$h(x, y) = \frac{2xy}{x + y}.$$

Properties of Rational Numbers

Theorem

The harmonic mean of rational numbers, when defined, is a rational number.

Properties of Rational Numbers

Proof.

- Let r and s be nonzero rational numbers.
- Let $r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some integers a, b, c, and d.
- Then a, b, c, and d are nonzero.

$$h(r,s) = \frac{1}{\left(\frac{\frac{1}{r}+\frac{1}{s}}{2}\right)} = \frac{2rs}{r+s}$$

$$= \frac{2 \cdot \frac{a}{b} \cdot \frac{c}{d}}{\frac{a}{b}+\frac{c}{d}} = \frac{2ac}{ad+bc}.$$

- And $\dfrac{2ac}{ad+bc}$ is a rational number provided $ad + bc \neq 0$.

□

Properties of Rational Numbers

Theorem

Let $0.d_1d_2d_3\ldots$ be the decimal representation of a real number a. If the representation is repeating, that is, if there exists an integer $n \geq 1$ and an integer $m \geq 1$ such that

$$d_{n+i} = d_{n+m+i}$$

for all $i \geq 0$, then a is rational.

- How would we prove it?
- Is the converse true?

Outline

Even Over Odd Rationals

Definition (Even Over Odd Rational)

A rational number $\frac{a}{b}$ is called even over odd if a is even and b is odd.

Theorem

Let r and s be even-over-odd rational numbers. Then $r + s$, $r - s$, and rs are also even-over-odd rational numbers.

- Is $\frac{r}{s}$ necessarily even over odd?
- Is it possible that $\frac{r}{x}$ is even over odd?

Odd Over Even Rationals

Definition (Odd Over Even Rational)

A rational number $\frac{a}{b}$ is called odd over even if a is odd and b is even.

- If r and s are odd-over-even rational numbers, then what can we say about $r + s$, $r - s$, rs, and $\frac{r}{s}$?

Outline

Intersecting Lines

Theorem

If two lines have distinct rational slopes and rational y-intercepts, then they intersect at a point with rational coordinates.

Solve the System

Proof.

- Let the lines be

$$L_1 : y = m_1 x + b_1,$$
$$L_2 : y = m_2 x + b_2,$$

  where m_1, m_2, b_1, and b_2 are rational and $m_1 \neq m_2$.
- Solve the equations simultaneously to find the point of intersection.

□

Solve the System

Proof.

- We get

$$x = \frac{b_2 - b_1}{m_1 - m_2},$$
$$y = \frac{b_1 m_2 - b_2 m_1}{m_1 - m_2}.$$

- Justify the claim that these are rational numbers.

□

Outline

Assignment

Assignment

- Read Section 4.2, pages 163 - 168.
- Exercises 5, 7, 8, 14, 16, 22, 27, 28, 30, 33, page 168.